# Sliding Theorem

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## Introduction

Scheduling constraints occasionally require the presence of some “feature” over all time intervals of fixed width. A typical example might be the following rule for rostering flight attendants:

> *Rosters of one month’s duration must have at least 24 consecutive hours free of duty among any 7 consecutive days*

Dividing the month into 720 hours, a roster may be represented by a sequence of 720 binary digits (0s and 1s), where 0 represents rest and 1 represents work.

Then this constraint may be expressed as:

> *Sequences of 720 binary digits must have at least 24 consecutive 0s among any consecutive 168 binary digits*

More generally, for any $1 \le L \le M \le N$:

> *Sequences of N binary digits must have at least L consecutive 0s among any consecutive M binary digits*

A SQL Server function for testing this rule on any binary sequence will be presented, using a few simple SELECTs on a row of integers. No looping or cursors are required because a little high school algebra does all the heavy lifting.

To understand the algebraic reasoning below, some definitions are required:

For any sequence $S = \langle s_i, i = 1,..., N\rangle$, its *length* len(S) is N.

If $S = \langle s_i, i = M,..., N\rangle$ is any sequence, its *starting position* is M.

If $S = \langle s_i, i = M,..., N\rangle$ and $T = \langle t_i, i = M1,..., N1\rangle$ are both sequences, then S is a *subsequence* of T, denoted $S \subseteq T$, if the following holds:

$$M1 \le M \le N \le N1$$

$$s_i = t_i \text{ for } i = M,..., N$$

In other words, subsequences are obtained from sequences by dropping some their beginning and ending elements. Clearly $\subseteq$ is a partial order.

If $S \subseteq T$, then S is said to be *contained* in T. To keep notation simple, subsequences S of a fixed sequence T will often be identified by their terminal indices:

$$[M,N] = \langle t_i,\ i = M,\dots, N\rangle.$$

A *work period* W is any sequence $\langle w_i,\ i = 1,\dots, N\rangle$ of binary digits $w_i$, $i = 1, \dots, N$.

Any subsequence of a work period containing just 0s is called a *rest period*. A rest period is *maximal* if it is not contained in a larger rest period.

In the following work period of length 22, some of the rest periods are shown, one per line. Those that are maximal are shaded.

| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | |
| 0 | | | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | | | | | | | | | | | | | | | | | | | |
| | 0 | 0 | | | | | | | | | | | | | | | | | | | |
| | | | | 0 | | | | | | | | | | | | | | | | | |
| | | | | | 0 | | | | | | | | | | | | | | | | |
| | | | | 0 | 0 | | | | | | | | | | | | | | | | |
| | | | | | | | | | 0 | | | | | | | | | | | | |
| | | | | | | | | | | | 0 | 0 | | | | | | | | | |

The following function defines the constraint for any W, N, M, L:

**Sliding(W,N,M,L)**

Return True if work period W of length N has every subsequence of length M containing at least L consecutive 0s. Otherwise, return False.

It assumes that $1 \le L \le M \le N$.

For example, Sliding(W,720,168,24) tests the constraint in the above example for any W.

Intuitively, it works as follows (although the actual computation is much different):

The function starts with the left-most subsequence $\langle w_i,\ i = 1,\dots, M\rangle$ of length M and determines if it contains at least L consecutive 0s. If successful, the subsequence "slides" to the right by one position $\langle w_i,\ i = 2,\dots, M+1\rangle$ and repeats the test (otherwise the test fails).

For a given W, N, M, L a maximal rest period is called *basic* if its length is at least L.

Clearly any pair of distinct basic periods are separated by at least one 1 (otherwise they wouldn't be maximal). Furthermore, basic periods are linearly ordered by their starting positions.

In the above example, if L = 2, the basic periods are those shown below:

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | | | | | | | | | | | | | | | | | | | |
| | | | | 0 | 0 | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | 0 | 0 | | | | | | | | | |
| | | | | | | | | | | | | | | | 0 | 0 | | | | | |
| | | | | | | | | | | | | | | | | | | | | 0 | 0 |

On the other hand, if L = 3, only the first maximal rest period is basic.

If L = 1, then they are all basic.

**Key Observation**

All non-basic rest periods may be removed by converting their 0s to 1 without affecting the constraint. That's because they'll never be used to verify the existence of L consecutive 0s since they have fewer than L 0s.

So, in the above sequence, the red 0s may be set to 1 before testing begins.

This reset is critical, since the Sliding theorem below assumes that adjacent rest periods are always basic.

It shows how the Sliding function can determine, using simple algebra, whether the constraint fails on any W, N, M, L passed to it.

To help understand the inequalities in the proof, it's handy to remember that

len([M,N]) = M - N + 1

for any subsequence [M,N]. In other words, the difference between two integers is one less than the number of integers between (or equal) to them.

**Sliding Theorem**

The constraint fails if and only (iff) at least one of the following holds:

(1) There are no basic periods

(2) The starting position s of the first basic period satisfies:

$$s \geq M - L + 2$$

(3) The ending position e of the last basic period satisfies:

$$e \leq N - M + L - 1$$

(4) The ending position e of some basic period and the starting position s of the next basic period satisfy:

$$s - e \geq M - 2L + 3$$

In other words, the constraint will fail iff a sufficiently large gap exists between a pair of adjacent basic periods, or the first one starts too late, or the last one ends too soon, or there aren't any basic periods.

Note that if conditions (2) or (3) hold for the first (or last) basic periods, then they hold for all of them.

By enumerating the position and length of all basic periods, the Sliding function can test each of the above conditions with a simple SELECT on the basic period positions and lengths.

**Proof**

We will show the left hand side implies the right, and vice versa.

LHS $\longrightarrow$ RHS

Suppose Sliding(W,N,M,L) fails.

Then there is a subsequence $S = [s_0,s_1]$ of length M that contains at most L - 1 consecutive 0s.

Assume there is at least one basic period. It remains to show that (2) or (3) or (4) holds.

Let $\mathcal{P}$ be the set of basic periods $P = [p_0,p_1]$ such that $p_0 < s_0$ and let $\mathcal{Q}$ be the set of basic periods $Q = [q_0,q_1]$ such that $s_0 \leq q_0$. Obviously, each basic period belongs to exactly one of $\mathcal{P}$ or $\mathcal{Q}$.

Since there is at least one basic period, exactly one of the following holds:

(a) $\mathcal{P}$ is non-empty but $\mathcal{Q}$ is empty

(b) $\mathcal{P}$ is empty but $\mathcal{Q}$ is non-empty

(c) $\mathcal{P}$ is non-empty and $\mathcal{Q}$ is non-empty

Suppose (a) holds.

Choose $P = [p_0,p_1] \in \mathcal{P}$.

Clearly $p_1 < s_1$ (otherwise S contains all 0s since $p_0 < s_0$).

Since at most L - 1 elements of P can belong to S:

$$p_1 - s_0 + 1 \leq L - 1$$

Note that this inequality holds even if $p_1 \leq s_0$ (and if $L = 1$ then $p_1 < s_0$).

Hence,

$$p_1 \leq s_0 - 1 + L - 1$$

But $s_0 + M - 1 = s_1 \leq N$ so $s_0 \leq N - M + 1$

Therefore,

$$p_1 \leq (N - M + 1) - 1 + L - 1 = N - M + L - 1$$

so (3) holds.

Suppose (b) holds. Choose $Q = [q_0,q_1] \in \mathcal{Q}$.

By definition of $\mathcal{Q}$, $s_0 \leq q_0$.

Since at most L - 1 elements of Q can belong to S:

$$s_1 - q_0 + 1 \leq L - 1$$

Note that this inequality holds even if $s_1 \leq q_0$ (and if $L = 1$ then $s_1 < q_0$).

Hence

$$q_0 \geq s_1 + 1 - L + 1 = s_0 + M - 1 + 1 - L + 1$$
$$\geq 1 + M - 1 + 1 - L + 1$$
$$= M - L + 2$$

so (2) holds.

Suppose (c) holds.

Choose $P = [p_0,p_1] \in \mathcal{P}$ with the largest starting point $p_0$ and $Q = [q_0,q_1] \in \mathcal{Q}$ with the smallest starting point $q_0$. Then P is the immediate predecessor of Q, as shown in Fig. 1:

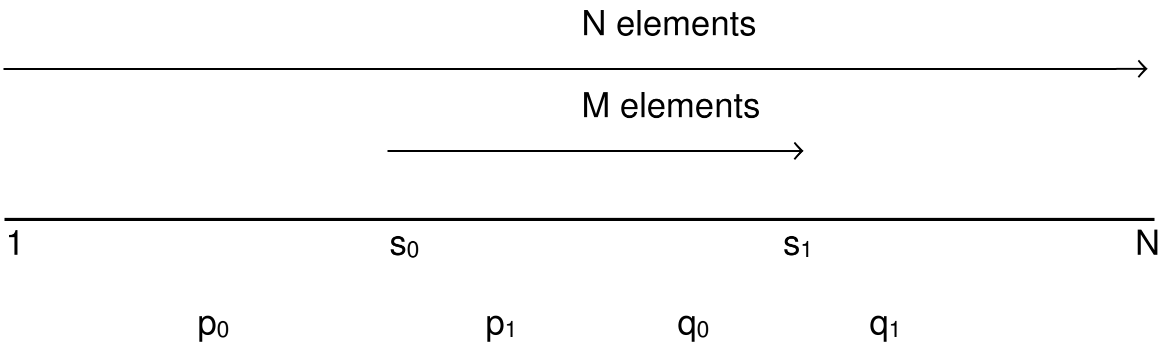


**Fig. 1**

Since the length of S is M:

$$s_1 - s_0 + 1 = M$$

By adding terms that cancel each other:

$$(p_1 - s_0) + (q_0 - p_1) + (s_1 - q_0) + 1 = M$$

$$(q_0 - p_1) = M - (p_1 - s_0) - (s_1 - q_0) - 1$$

Since there can be at most L - 1 elements between $s_0$ and $p_1$:

$$p_1 - s_0 + 1 \leq L - 1$$
$$p_1 - s_0 \leq L - 1 - 1$$
$$p_1 - s_0 \leq L - 2$$

Note that if $p_1 < s_0$ the inequality holds trivially.

For similar reasons:

$$s_1 - q_0 + 1 \leq L - 1$$
$$s_1 - q_0 \leq L - 1 - 1$$
$$s_1 - q_0 \leq L - 2$$

Combined with above equality we derive:

$$q_0 - p_1 \geq M - (L - 2) - (L - 2) - 1 = M - 2L + 3$$

so (4) follows.

RHS $\longrightarrow$ LHS

If (1) holds then Sliding(W,N,M,L) immediately fails.

Suppose (2) holds for the first basic period $P = [p_o,p_1]$:

$$p_o \geq M - L + 2$$

$$M - p_0 + 1 \leq L - 1$$

Consider $S = [1,M]$, which is a subsequence of length M.

Let n be the number of members of P contained in S.

Then S will contain L consecutive 0s iff $n \geq L$ since P is the first basic period.

But

$$n \leq \max\{0, M - p_0 + 1\} \leq \max\{0, L - 1\}$$

Note that this inequality holds even if $M \leq p_0$ (and if $L = 1$ then $M < p_0$).

For this is follows that S cannot contain L consecutive 0s.

Hence Sliding(W,N,M,L) fails.

Suppose (3) holds for the last basic period $Q = [q_0,q_1]$:

$$q_1 \leq N - M + L - 1$$

$$q_1 - N + M \leq L - 1$$

$$q_1 - (N - M + 1) + 1 \leq L - 1$$

Consider $S = [N - M + 1, N]$, which is a subsequence of length M.

Let n be the number of members of Q contained in S.

Then S will contain L consecutive 0s iff $n \geq L$ since Q is the last basic period.

But

$$n \le \max\{0, q_1 - (N - M + 1) + 1\} \le \max\{0, L - 1\}$$

Note that this inequality holds even if $N - M + 1 \ge q_1$ (and if $L = 1$ then $N - M + 1 > q_1$).

So, S cannot contain L consecutive 0s.

Hence Sliding(W,N,M,L) fails.

Suppose (4) holds for a basic period $P = [p_0,p_1]$ and the next basic period $Q = [q_o,q_1]$:

$$q_0 - p_1 \ge M - 2L + 3.$$

Consider $S = [p_1 - L + 1, q_0 + L - 1]$, which consists of the $q_0 - p_1 - 1$ elements strictly between $p_1$ and $q_0$ joined with the last L elements of P and the first L elements of Q.

The length of S is:

$$\begin{aligned} q_0 + L - 1 - (p_1 - L + 1) + 1 &= q_0 - p_1 + 2L - 1 \\ &\ge M - 2L + 3 + 2L - 1 \\ &= M + 2. \end{aligned}$$

By deleting its first and last members, S is reduced to a subsequence of length M containing L - 1 consecutive 0s, but no more.

Hence Sliding(W,N,M,L) fails.